

Discrete symmetries in the Kaluza-Klein-like theories

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In theories of the Kaluza-Klein type there are spins (or rather total angular momentum) in higher dimensions which manifest as charges in the observable $d = (3 + 1)$. The charge conjugation requirement, if following the prescription in $(3 + 1)$, would transform any particle state out of the Dirac sea into the hole in the Dirac sea, which manifests as an anti-particle having all the spin degrees of freedom in d , except S^{03} , the same as the corresponding particle state. This is in contradiction with what we observe for the anti-particle. In this paper we redefine the discrete symmetries so that we stay within the subgroups of the starting group of symmetries, while we require that the spin in higher dimensions manifest as charges in $d = (3 + 1)$. We pay attention on spaces with $d = 2 \pmod{4}$.

I. INTRODUCTION

Discrete symmetries C , P and T play an important role in physics. As is well known, the product of all three symmetries seems to be conserved, while C and P and even the product of C and P , and correspondingly T itself, are not conserved. The non conservation of CP is well measured in kaons, and manifests in cosmology also in the matter anti-matter asymmetry.

In the Kaluza-Klein-like theories [1–3] the total angular moments, if there is rotational invariance (at all), in higher dimensions manifests as the charges in $(3 + 1)$. In the low energy regime it is indeed the spin degrees of freedom, which manifest as the conserved charges [4]. One of the authors of this paper (NSMB) is proposing the theory, named the *spin-charge-family* theory [5, 7, 8, 10], which by offering the mechanism for the generation of families predicts the number of observable families at low energies. The theory is a kind of the Kaluza-Klein-like theory from the point of view that the spins (angular moments) in higher dimensions manifest as charges in $d = (3 + 1)$. Both authors have published together several papers, proving that in non-compact spaces the break of the starting symmetry in d may allow massless fermions after the break [4]. Discrete symmetries are very important part of any theory: They must be in agreement with the experiments.

Extending the prescription of the discrete symmetries from $d = (3 + 1)$ to any d , the anti-particle

to a chosen particle would have in the second quantized theory all the components of spin (except the S^{03} component, since in quantum mechanics time is a parameter) the same as a particle, which means that it would have all the charges the same as the corresponding particle. This would be in contradiction with what we observe, namely that the anti-particle to a chosen particle has opposite charges.

In this paper we present the redefinition of the discrete symmetries – defined within the subgroups of the starting group of symmetries – which offers experimentally observed properties of anti-particles in $d = (3 + 1)$. We pay attention on spaces with $d = 2 \pmod{4}$, the dimensions in which particles interacting only gravitationally and carrying only spin are mass protected [9]. We do not study in this paper the break of the CP and correspondingly of the T symmetry. We namely do not include into this study the break of symmetries causing phase transitions.

II. DISCRETE SYMMETRIES FOR FREE FERMIONS IN D-DIMENSIONS FOLLOWING THE DEFINITIONS IN $d = (3 + 1)$

We start with the definition of the discrete symmetries as they follow from the prescription in $d = (3 + 1)$.

We define the $\mathcal{C}_{\mathcal{H}}$ operator to be distinguished from the $\mathbb{C}_{\mathcal{H}}$ operator. The first transforms any single particle state Ψ_i^p , which solves the Weyl equation for a positive energy and it is in the second quantized theory understood as the state above the Dirac sea, into the charge conjugate one with the negative energy Ψ_i^n and correspondingly belonging to a state in the Dirac sea

$$\mathcal{C}_{\mathcal{H}} = \prod_{\gamma^a \in \mathfrak{S}} \gamma^a K. \quad (1)$$

The product of the imaginary γ^a operators is meant in ascending order. We make a choice of γ^0, γ^1 real, γ^2 imaginary, γ^3 real, γ^5 imaginary, γ^6 real, and alternating real and imaginary ones we end up in even dimensional spaces with real γ^d . K makes complex conjugation, transforming i into $-i$.

$\mathbb{C}_{\mathcal{H}}$ empties this negative energy state in the Dirac sea creating an anti-particle with the positive energy and all the properties of the starting single particle state above the Dirac sea, that is with the same d -momentum and the same all the spin degrees of freedom, except the S^{03} value (since in quantum mechanics the Hilbert space is defined at a particular (any) time, while we follow a particle (or an anti-particle) in the time evolution through states parameterised by the time), as the starting single particle state. Let Ψ^p be the destruction operator annihilating a particle in the

state $\Psi(\vec{x})$.

$$\begin{aligned}\Psi^{p\dagger}|vac> &= \left(\int \Psi^{p\dagger}(x) \Psi^p(x) d^{d-1}x\right) |vac>, \\ (\mathcal{C}_{\mathcal{H}} \Psi^{p\dagger} &= \int \Psi^{p\dagger}(\vec{x}) (\mathcal{C}_{\mathcal{H}} \Psi^p(\vec{x})) d^{d-1}x) |vac>.\end{aligned}\quad (2)$$

We define the time reversal operator $\mathcal{T}_{\mathcal{H}}$ and the parity operator $\mathcal{P}_{\mathcal{H}}$ as follows

$$\begin{aligned}\mathcal{T}_{\mathcal{H}} &= \prod_{\gamma^a \in \mathbb{R}} \gamma^a K I_t, \quad a \neq 0, \quad I_t x^0 \rightarrow -x^0, \quad I_t x^a \rightarrow -x_a, \\ \mathcal{P}_{\mathcal{H}}^{(d-1)} &= \gamma^0 I_{\vec{x}}, \quad I_x \vec{x} \rightarrow -\vec{x} \quad I_{\vec{x}} x^a \rightarrow x_a.\end{aligned}\quad (3)$$

Again is the product $\prod \gamma^a$ meant in the ascending order in γ^a .

We shall look for solutions of the Weyl equation in $d = (d-1) + 1$ for $d = 2 \pmod{4} = 2(2n+1)$, $n = 1, 2, \dots$

$$\gamma^a p_a \psi = 0, \quad (4)$$

and show the application of the above defined discrete symmetries on the solutions for two particular cases: **i.** $d = (5+1)$, the properties of which we study in several papers [4], and **ii.** $d = (13+1)$, which one of the authors of this paper uses in her *spin-charge-family* theory, since it manifests in $d = (3+1)$ in the low energy regime the family members with all their symmetry properties assumed by the *standard model* (extended with the right handed neutrino) and explains the appearance of families. Both spaces belong to the spaces with $d = 2 \pmod{4}$. Let us recognize that the operator of handedness, expressed in terms of the Cartan sub algebra members, is as follows

$$\Gamma^{((d-1)+1)} = (-2i)^{\frac{d}{2}} S^{03} S^{12} S^{56} \dots S^{(d-1)d}. \quad (5)$$

For the choice of the coordinate system so that d -momentum manifests $p^a = (p^0, 0, 0, p^3, 0 \dots 0)$ the Weyl equation simplifies to

$$(-2i S^{03} p^0 = p^3) \psi. \quad (6)$$

We shall make use of this choice. Solutions in the coordinate representation are plane waves: $e^{-ip^a x_a}$. In this part $\mathcal{T}_{\mathcal{H}}$ and $\mathcal{P}_{\mathcal{H}}$ manifest as follows

$$\mathcal{T}_{\mathcal{H}} e^{-ip^0 x^0 + ip^3 x^3} = e^{-ip^0 x^0 - ip^3 x^3}, \quad \mathcal{P}_{\mathcal{H}} e^{-ip^0 x^0 + ip^3 x^3} = e^{-ip^0 x^0 - ip^3 x^3}, \quad (7)$$

since in the momentum representation only p^a is a vector, while x^a is just a parameter and opposite in the coordinate representation.

Let us now demonstrate the application of the discrete operators $\mathbb{C}_{\mathcal{H}}$, $\mathcal{T}_{\mathcal{H}}$ and $\mathcal{P}_{\mathcal{H}}$ on one Weyl representation from Table I representing the positive and negative energy solutions of the Weyl equation (6) in $d = (5 + 1)$. Here and in what follows we do not pay attention on the normalization factor of the single particle states. Let us make a choice of the positive energy state $\psi_1^p = \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)} e^{-ip^0x^0+ip^3x^3}$, for example. We use the technique of the refs. [15]. A short overview can be found in the appendix. The reader is kindly asked to look for more detailed explanation in [15]. It follows for $p^0 = |p^0|$ and $p^3 = |p^3|$

$$\mathcal{C}_{\mathcal{H}}\psi_1^p \rightarrow \overset{03}{(+i)}\overset{12}{[-]}\overset{56}{[-]} e^{ip^0x^0-ip^3x^3} = \psi_2^n. \quad (8)$$

This state is the solution of the Weyl equation for the negative energy state. But the hole of this state in the Dirac sea makes a positive energy state (above the Dirac sea) with the properties of the starting state, but it is an anti-particle state: $\Psi^a = \overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)} e^{-ip^0x^0+ip^3x^3}$, defined on the Dirac sea with the hole belonging to the negative energy single-particle state ψ_2^n . Namely, $\mathbb{C}_{\mathcal{H}}\Psi^p\mathbb{C}_{\mathcal{H}}^{-1}$, when applies on the vacuum state, represents an anti-particle.

This anti-particle state is correspondingly the solution of the same Weyl equation, belongs to the same representation as the starting state and it is obviously a good symmetry in $d = 2 \pmod{4}$ spaces. But this state has the S^{56} spin, which should represent the charge of the anti-particle in $d = (3 + 1)$, the same as the starting state. This is not in agreement with what we observe.

Since either $\mathcal{T}_{\mathcal{H}} (\mathcal{T}_{\mathcal{H}}\psi_1^p = \overset{03}{[-i]}\overset{12}{[-]}\overset{56}{[-]} e^{-ip^0x^0-ip^3x^3})$ or $\mathcal{P}_{\mathcal{H}} (\mathcal{P}_{\mathcal{H}}\psi_1^p = \overset{03}{[-i]}\overset{12}{(+)}\overset{56}{(+)} e^{-ip^0x^0-ip^3x^3})$ is defined with an odd number of γ^a operators, none of them are the symmetry (the conserved operators) within one Weyl representation, since both transform correspondingly the starting state into a state of another Weyl representation. This is true for all the spaces with $d = 2 \pmod{4}$.

The product of $\mathcal{T}_{\mathcal{H}}$ and $\mathcal{P}_{\mathcal{H}}^{(d-1)}$ is again a good symmetry, transforming the starting state, say ψ_1^p , into a positive energy state of the same Weyl representation

$$\mathcal{T}_{\mathcal{H}}\mathcal{P}_{\mathcal{H}}^{(d-1)}\psi_1^p = \overset{03}{(+i)}\overset{12}{[-]}\overset{56}{[-]} e^{-ip^0x^0+ip^3x^3} = \psi_2^p, \quad (9)$$

and solving the Weyl equation.

Also the product of all three discrete symmetries is correspondingly a good symmetry as well, transforming the starting state (put on the top of the Dirac sea) into the positive energy anti-particle state

$$\mathbb{C}_{\mathcal{H}}\mathcal{T}_{\mathcal{H}}\mathcal{P}_{\mathcal{H}}^{(d-1)}\psi_1^p \rightarrow \psi_2^p, \quad (10)$$

which is the hole in the state ψ_1^n in the Dirac sea.

Let us now look at the (a more) realistic case, that is at $d = (13 + 1)$ case, the positive energy states of which are presented in Table II. Following the procedure used in the previous case of $d = (5 + 1)$, the operator $\mathcal{C}_\mathcal{H}$ transforms, let say, the first state in Table II, which represents due to its quantum numbers the right handed (with respect to $d = (3 + 1)$) u -quark with spin up, weak chargeless, carrying the colour charge $(\frac{1}{2}, \frac{1}{(2\sqrt{3})})$, the second $SU(2)_{II}$ charge $\frac{1}{2}$, the hyper charge $\frac{2}{3}$ and the electromagnetic charge $\frac{2}{3}$, while it carries the momentum $p^a = (p^0, 0, 0, p^3, 0, \dots, 0)$, as follows

$$\mathcal{C}_\mathcal{H} u_{1R} = \begin{matrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i)[-] & [-] & [-] & [-] & || & [-] & [+] & [+] & \end{matrix} e^{ip^0 x^0 - ip^3 x^3}. \quad (11)$$

This state solves the Weyl equation for the negative energy and inverse momentum, carrying all the eigen values of the Cartan sub algebra operators $(S^{12}, S^{56}, S^{78}, S^{910}, S^{1112}, S^{1314})$, except S^{03} , of the opposite values than the starting state (this negative energy state is a part of the starting Weyl representation, not presented in Table II, but the reader can find this state in the ref. [6]). The second quantized charge conjugation operator $\mathbb{C}_\mathcal{H}$ empties this state in the Dirac sea, creating the anti-particle state to the starting state with all the quantum numbers of the starting state, obviously in contradiction with the observations, that the anti-particle state has the same momentum in $d = (3 + 1)$ but opposite charges than the starting state.

We conclude that *the second quantized anti-particle state* (the hole in the Dirac sea) *manifests correspondingly all the quantum numbers of the starting state, but it is the anti-particle*. Assuming that the eigen values of the Cartan sub algebra members in $d \geq 5$ represent charges in $d = (3 + 1)$, the *charges should have opposite values*, which the definition of the discrete symmetries operators in Eqs. (1, 2, 3) does not offer. The charge conjugation operation is a good symmetry in any $d = 2 \pmod{4}$ from the point of view that in any of spaces with $d = 2 \pmod{4}$ $\mathcal{C}_\mathcal{H} \psi_i^p$ defines the state within the same Weyl representation due to the fact that it is defined as the product of an even number of imaginary operators γ^a .

The product of the time reversal and the parity operation is in the space with $d = 2 \pmod{4} = 2(2n + 1)$ again a good symmetry, which means that it transforms the starting state of a chosen Weyl representation into the state belonging to the same Weyl representation, with the same d -momentum as the starting state.

ψ_i^p	positive energy state			$\frac{p^0}{ p^0 }$	$\frac{p^3}{ p^3 }$	$(-2iS^{03})$	$\Gamma^{(3+1)}$
ψ_1^p	$\overset{03}{(+i)} \overset{12}{(+)} \overset{56}{(+)}$		$e^{-ip^0x^0+ip^3x^3}$	+1	+1	+1	+1
ψ_2^p	$\overset{03}{(+i)} \overset{12}{[-]} \overset{56}{[-]}$		$e^{-ip^0x^0+ip^3x^3}$	+1	+1	+1	-1
ψ_3^p	$\overset{03}{[-i]} \overset{12}{[-]} \overset{56}{(+)}$		$e^{-ip^0x^0-ip^3x^3}$	+1	-1	-1	+1
ψ_4^p	$\overset{03}{[-i]} \overset{12}{(+)} \overset{56}{[-]}$		$e^{-ip^0x^0-ip^3x^3}$	+1	-1	-1	-1
ψ_i^n	negative energy state			$\frac{p^0}{ p^0 }$	$\frac{p^3}{ p^3 }$	$(-2iS^{03})$	$\Gamma^{(3+1)}$
ψ_1^n	$\overset{03}{(+i)} \overset{12}{(+)} \overset{56}{(+)}$		$e^{ip^0x^0-ip^3x^3}$	-1	-1	+1	+1
ψ_2^n	$\overset{03}{(+i)} \overset{12}{[-]} \overset{56}{[-]}$		$e^{ip^0x^0-ip^3x^3}$	-1	-1	+1	-1
ψ_3^n	$\overset{03}{[-i]} \overset{12}{[-]} \overset{56}{(+)}$		$e^{ip^0x^0+ip^3x^3}$	-1	+1	-1	+1
ψ_4^n	$\overset{03}{[-i]} \overset{12}{(+)} \overset{56}{[-]}$		$e^{ip^0x^0+ip^3x^3}$	-1	+1	-1	-1

TABLE I: Four positive energy states and four negative energy states, the solutions of Eq. (6), half have $\frac{p^3}{|p^3|}$ positive and half negative. $p^a = (p^0, 0, 0, p^3, 0, 0)$, $p^0 = |p^0|$ and $p^3 = |p^3|$, $\Gamma^{(5+1)} = -1$. Nilpotents $(k)^{ab}$ and projectors $[k]^{ab}$ operate on the vacuum state $|vac\rangle$ not written in the table.

A. Solutions of the Weyl equations in $d = (5 + 1)$

There are $2^{\frac{d}{2}-1} = 4$ basic spinor states of one family representation in $d = (5 + 1)$. (And there are $2^{\frac{d}{2}-1} = 4$ families of spinors.) Since the operators of Eqs. (1, 3) do not distinguish among the families, all the families behave equivalently with respect to these discrete symmetry operators. One of the family representation with four basic states is in the technique [15], described in terms of nilpotents $(k)^{ab}$ and projectors $[k]^{ab}$ (see appendix), as follows

$$\begin{aligned}
\Psi_1 &= \overset{03}{(+i)} \overset{12}{(+)} \overset{56}{(+)} |vac\rangle, \\
\Psi_2 &= \overset{03}{(+i)} \overset{12}{[-]} \overset{56}{[-]} |vac\rangle, \\
\Psi_3 &= \overset{03}{[-i]} \overset{12}{[-]} \overset{56}{(+)} |vac\rangle, \\
\Psi_4 &= \overset{03}{[-i]} \overset{12}{(+)} \overset{56}{[-]} |vac\rangle.
\end{aligned} \tag{12}$$

All the basic states are eigen states of the Cartan sub algebra, for which we take: S^{03}, S^{12}, S^{56} , with the eigen values, which can be read from Eq. (12) if taking $\frac{1}{2}$ of the numbers $\pm i$ or ± 1 in the parentheses () (nilpotents) and [] (projectors). We look for the solutions of Eq. (6) for a particular choice of the d -momentum $p^a = (p^0, 0, 0, p^3, 0, 0)$, $p^0 = |p^0|$ and $p^3 = |p^3|$, and find what is presented in Table I.

B. Solutions of the Weyl equations in $d = (13 + 1)$

There are $2^{\frac{d}{2}-1} = 64$ basic spinor states of one family representation in $d = (13 + 1)$. We again do not pay attention on the families, since all behave equivalently with respect to the discrete symmetries presented in Eqs. (1, 3). We present in this subsection positive energy states for quarks of a particular charge ($\tau^{33} = 1/2$, $\tau^{38} = 1/(2\sqrt{3})$) and for the chargeless leptons. The solutions for, say, the right handed u -quark, u_{1R} , with spin up, with the colour charge ($\tau^{33} = -1/2$ and $\tau^{38} = 1/(2\sqrt{3})$), weak chargeless and with a positive momentum p^3 is proportional to $((+i) \begin{smallmatrix} 03 & 12 & 56 & 78 \\ (+) & (+) & (+) & (+) \end{smallmatrix} \parallel \begin{smallmatrix} 9 & 10 & 11 & 12 \\ (+) & (-) & (-) & (-) \end{smallmatrix} e^{-ip^0x^0+ip^3x^3})$, while with the colour charge ($\tau^{33} = 0$ and $\tau^{38} = -1/\sqrt{3}$) is proportional to $((+i) \begin{smallmatrix} 03 & 12 & 56 & 78 \\ (+) & (+) & (+) & (+) \end{smallmatrix} \parallel \begin{smallmatrix} 9 & 10 & 11 & 12 \\ (+) & (+) & (-) & (-) \end{smallmatrix} e^{-ip^0x^0+ip^3x^3})$. The last state follows from the state of the first colour just by the application of the generators τ^{3i} of the colour group $SU(3)$. The definition of the generators of the subgroups expressed as the superposition of S^{ab} can be found in Table II. The lepton states follow from the quark states by transforming the colour charge part $\parallel \begin{smallmatrix} 9 & 10 & 11 & 12 \\ (+) & (-) & (-) & (-) \end{smallmatrix}$ into $\parallel \begin{smallmatrix} 9 & 10 & 11 & 12 \\ (+) & (+) & (+) & (+) \end{smallmatrix}$. (The negative energy solutions of the Weyl equation can be found, equivalently to the $d = (5 + 1)$ case, by the application of S^{ab} which do not belong to the Cartan sub algebra.)

The lepton representation of the $SO(7,1)$ subgroup of the $SO(13,1)$ group can be found in Table III.

III. DISCRETE SYMMETRIES IN $d = 2 \pmod{4}$ WITH THE DESIRED PROPERTIES IN $d = (3 + 1)$

In section II we define the discrete symmetries in spaces $d > (3 + 1)$ as they follow from the definition in $d = (3 + 1)$. This definition, however, does not allow to interpret the angular momentum (which at the low energy regime manifests as the spin) in higher than four dimensions as charges in $(3 + 1)$. The proposed charge conjugated states have, namely, the same charges as the starting states.

We look for new discrete symmetries, which would lead to the desired properties of the anti-particle state to any second quantized state:

- i. The anti-particle state has the same momentum in $(d = (3 + 1))$ as the starting state.
- ii. The anti-particle state has the same total angular momentum $J_{mn}, (m, n) \in (1, 2, 3)$ as the starting state.
- iii. The anti-particle state has the opposite values of the Cartan subalgebra of the total momentum

ψ_i^p	positive energy state	$\frac{p^0}{ p^0 }$	$\frac{p^3}{ p^3 }$	$(-2iS^{03})$	$\Gamma^{(3+1)}$	τ^{13}	τ^{23}	τ^4	Y	Q
u_{1R}	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9\ 10 & 11\ 12 & 13\ 14 \\ (+i) & (+) & & (+) & (+) & & (+) & (-) & (-) \end{smallmatrix} e^{-ip^0x^0+ip^3x^3}$	+1	+1	+1	+1	0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
u_{2R}	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9\ 10 & 11\ 12 & 13\ 14 \\ [-i] & [-] & & (+) & (+) & & (+) & (-) & (-) \end{smallmatrix} e^{-ip^0x^0-ip^3x^3}$	+1	-1	-1	+1	0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
d_{1R}	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9\ 10 & 11\ 12 & 13\ 14 \\ (+i) & (+) & & [-] & [-] & & (+) & (-) & (-) \end{smallmatrix} e^{-ip^0x^0+ip^3x^3}$	+1	+1	+1	+1	0	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
d_{2R}	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9\ 10 & 11\ 12 & 13\ 14 \\ [-i] & [-] & & [-] & [-] & & (+) & (-) & (-) \end{smallmatrix} e^{-ip^0x^0-ip^3x^3}$	+1	-1	-1	+1	0	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
d_{1L}	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9\ 10 & 11\ 12 & 13\ 14 \\ [-i] & (+) & & [-] & (+) & & (+) & (-) & (-) \end{smallmatrix} e^{-ip^0x^0-ip^3x^3}$	+1	-1	-1	-1	$-\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
d_{2L}	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9\ 10 & 11\ 12 & 13\ 14 \\ (+i) & [-] & & [-] & (+) & & (+) & (-) & (-) \end{smallmatrix} e^{-ip^0x^0+ip^3x^3}$	+1	+1	+1	-1	$-\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
u_{1L}	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9\ 10 & 11\ 12 & 13\ 14 \\ [-i] & (+) & & (+) & [-] & & (+) & (-) & (-) \end{smallmatrix} e^{-ip^0x^0-ip^3x^3}$	+1	-1	-1	-1	$\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
u_{2L}	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9\ 10 & 11\ 12 & 13\ 14 \\ (+i) & [-] & & (+) & [-] & & (+) & (-) & (-) \end{smallmatrix} e^{-ip^0x^0+ip^3x^3}$	+1	+1	+1	-1	$\frac{1}{2}$	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$

TABLE II: One $SO(7, 1)$ sub representation of the representation of $SO(13, 1)$, the one representing quarks, which carry the colour charge ($\tau^{33} = 1/2$, $\tau^{38} = 1/(2\sqrt{3})$). All members have $\Gamma^{(13+1)} = -1$. All states are the eigen states of the Cartan sub algebra ($S^{03}, S^{12}, S^{56}, S^{7,8}, S^{9\ 10}, S^{11\ 12}, S^{13\ 14}$) and solve the Weyl equation (6) for the choice of the coordinate system $p^a = (p^0, 0, 0, p^3, 0, \dots, 0)$. The infinitesimal generators of the weak charge $SU(2)_W = SU(2)_I$ group are defined as ($\vec{\tau}^1 = \frac{1}{2}(S^{58} - S^{67}, S^{57} + S^{6,8}, S^{56} - S^{7,8})$), of another $SU(2)_{II}$ as ($\vec{\tau}^2 = \frac{1}{2}(S^{58} + S^{6,7}, S^{57} - S^{6,8}, S^{56} + S^{7,8})$), of the τ^4 charge as ($-\frac{1}{3}(S^{9\ 10} + S^{11\ 12} + S^{13\ 14})$) and of the colour charge group as ($\vec{\tau}^3 = (\frac{1}{2}(S^{9\ 12} - S^{10\ 11}, S^{9\ 11} + S^{10\ 12}, S^{9\ 10} - S^{11\ 12}, S^{9\ 14} - S^{10\ 13}, S^{9\ 13} + S^{10\ 14}, S^{11\ 14} - S^{12\ 13}, S^{11\ 13} + S^{12\ 14}, \frac{1}{\sqrt{3}}(S^{9\ 10} + S^{11\ 12} - 2S^{13\ 14}))$), $Y = \tau^{23} + \tau^4$, $Q = \tau^{13} + Y$. Nilpotents $\overset{ab}{(k)}$ and projectors $\overset{ab}{[k]}$ operate on the vacuum state $|vac\rangle$ not written in the table.

ψ_i^p	positive energy state	$\frac{p^0}{ p^0 }$	$\frac{p^3}{ p^3 }$	$(-2iS^{03})$	$\Gamma^{(3+1)}$	τ^{13}	τ^{23}	τ^4	Y	Q
ν_{1R}	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9\ 10 & 11\ 12 & 13\ 14 \\ (+i) & (+) & & (+) & (+) & & (+) & [+] & [+] \end{smallmatrix} e^{-ip^0x^0+ip^3x^3}$	+1	+1	+1	+1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
ν_{2R}	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9\ 10 & 11\ 12 & 13\ 14 \\ [-i] & [-] & & (+) & (+) & & (+) & [+] & [+] \end{smallmatrix} e^{-ip^0x^0-ip^3x^3}$	+1	-1	-1	+1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
e_{1R}	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9\ 10 & 11\ 12 & 13\ 14 \\ (+i) & (+) & & [-] & [-] & & (+) & [+] & [+] \end{smallmatrix} e^{-ip^0x^0+ip^3x^3}$	+1	+1	+1	+1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	-1
e_{2R}	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9\ 10 & 11\ 12 & 13\ 14 \\ [-i] & [-] & & (+) & (+) & & (+) & [+] & [+] \end{smallmatrix} e^{-ip^0x^0-ip^3x^3}$	+1	-1	-1	+1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	-1
e_{1L}	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9\ 10 & 11\ 12 & 13\ 14 \\ [-i] & (+) & & [-] & (+) & & (+) & [+] & [+] \end{smallmatrix} e^{-ip^0x^0-ip^3x^3}$	+1	-1	-1	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
e_{2L}	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9\ 10 & 11\ 12 & 13\ 14 \\ (+i) & [-] & & [-] & (+) & & (+) & [+] & [+] \end{smallmatrix} e^{-ip^0x^0+ip^3x^3}$	+1	+1	+1	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
ν_{1L}	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9\ 10 & 11\ 12 & 13\ 14 \\ [-i] & (+) & & (+) & [-] & & (+) & [+] & [+] \end{smallmatrix} e^{-ip^0x^0-ip^3x^3}$	+1	-1	-1	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
ν_{2L}	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9\ 10 & 11\ 12 & 13\ 14 \\ (+i) & [-] & & (+) & [-] & & (+) & [+] & [+] \end{smallmatrix} e^{-ip^0x^0+ip^3x^3}$	+1	+1	+1	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0

TABLE III: One $SO(7, 1)$ sub representation of the representation of $SO(13, 1)$, the one representing colour chargeless leptons. All members have, as in the quark case, $\Gamma^{(13+1)} = -1$. All states are the eigen states of the Cartan sub algebra ($S^{03}, S^{12}, S^{56}, S^{7,8}, S^{9\ 10}, S^{11\ 12}, S^{13\ 14}$) and solve the Weyl equation (6) for the choice of the coordinate system $p^a = (p^0, 0, 0, p^3, 0, \dots, 0)$, just as in the quark case. The rest of notation is the same as in Table II.

$J_{s,t} = L^{st} + S^{st}$, $(s,t) \in (5,6,\dots,d)$, or at low energies rather the opposite values of the Cartan subalgebra of S^{st} , $(s,t) \in (5,6,\dots,d)$ as the starting state.

The manifestation of the spin degrees of freedom in $d > 4$ as charges in $d \leq 4$ depends on the symmetries that the (non-)compact spaces manifest [4]. For the toy model [4] in $d = (5+1)$ the spin on the infinite disc (curled into an almost a sphere) manifests for a massless spinor as a charge in $d = (3+1)$. Only to the massive states the angular momentum in $d = (5,6)$ contributes.

In the case of the *spin-charge-family* theory, which manifests at low energies effectively as the *standard model*, the operators $\bar{\tau}^1, \bar{\tau}^2, \bar{\tau}^3, Y, \tau^4, Q$ (which all are superposition of S^{ab} , $a, b \in \{5,6,\dots,14\}$) define charges in $d = (3+1)$.

We define new discrete symmetries by transforming the above defined discrete symmetries ($\mathbb{C}_{\mathcal{H}}$, $\mathcal{T}_{\mathcal{H}}$ and $\mathcal{P}_{\mathcal{H}}$) so that, while remaining within the same groups of symmetries, the redefined discrete symmetries manifest the experimentally acceptable properties in $d = (3+1)$, which is of the essential importance for all the Kaluza-Klein-like theories [2] without any degrees of freedom of fermions besides the spin and family quantum numbers [5, 8]. We define new discrete symmetries as follows

$$\begin{aligned}\mathbb{C}_{\mathcal{N}} &= \mathbb{C}_{\mathcal{H}} \mathcal{P}_{\mathcal{H}}^{(d-1)} e^{i\pi J_{12}} e^{i\pi J_{35}} e^{i\pi J_{79}} e^{i\pi J_{1113}}, \dots, e^{i\pi J_{(d-3)(d-1)}}, \\ \mathcal{T}_{\mathcal{N}} &= \mathcal{T}_{\mathcal{H}} \mathcal{P}_{\mathcal{H}}^{(d-1)} e^{i\pi J_{12}} e^{i\pi J_{36}} e^{i\pi J_{810}} e^{i\pi J_{1214}}, \dots, e^{i\pi J_{(d-2)d}}, \\ \mathcal{P}_{\mathcal{N}}^{(d-1)} &= \mathcal{P}_{\mathcal{H}}^{(d-1)} e^{i\pi J_{56}} e^{i\pi J_{78}} e^{i\pi J_{910}} e^{i\pi J_{1112}} e^{i\pi J_{1314}}, \dots, e^{i\pi J_{(d-1)d}}.\end{aligned}\tag{13}$$

The rotations ($e^{i\pi J_{12}} e^{i\pi J_{35}} e^{i\pi J_{79}} \dots, e^{i\pi J_{(d-3)(d-1)}}$) together with $\mathcal{P}_{\mathcal{H}}^{(d-1)}$, which are included in $\mathbb{C}_{\mathcal{N}}$, keep p^i for $i = (1,2,3)$ unchanged, while they transform a state so that all the eigen values of the Cartan sub algebra except (S^{03}, S^{12}) change sign. Correspondingly this redefined $\mathbb{C}_{\mathcal{N}}$ transforms a second quantized state into the antiparticle state with the same four momentum as the starting state but with the opposite values of the spin determined by the Cartan sub algebra eigen values, except for S^{03} and S^{12} . Parity operator $\mathcal{P}_{\mathcal{N}}^{(d-1)}$ changes p^i into $-p^i$ only for $i = (1,2,3)$, while the time reversal operator corrects all the properties of the new $\mathbb{C}_{\mathcal{N}}$ and $\mathcal{P}_{\mathcal{N}}^{(d-1)}$ so that

$$\mathbb{C}_{\mathcal{N}} \mathcal{P}_{\mathcal{N}}^{(d-1)} \mathcal{T}_{\mathcal{N}} = \mathbb{C}_{\mathcal{H}} \mathcal{P}_{\mathcal{H}}^{(d-1)} \mathcal{T}_{\mathcal{H}}.\tag{14}$$

All three new operators commute among themselves as also the old ones do.

Let us now see on two cases, for $d = (5+1)$ and for $d = (13+1)$, how do the new proposals for the discrete symmetries, $\mathbb{C}_{\mathcal{N}}$, $\mathcal{P}_{\mathcal{N}}^{(d-1)}$, $\mathcal{T}_{\mathcal{N}}$, manifest.

Charge conjugation symmetry $\mathbb{C}_{\mathcal{N}}$:

Let us start with ψ_1^p from Table I. In $d = (5 + 1)$ the charge conjugation operator $\mathbb{C}_\mathcal{N}$ equals to $\mathbb{C}_\mathcal{H} \mathcal{P}_\mathcal{H}^{(d-1)} e^{i\pi J_{12}} e^{i\pi J_{35}}$. To test this symmetry on the second quantized state Ψ_1^p one can start with Eq. (8) and the recognition below this equation that $\mathbb{C}_\mathcal{H}$ transforms a second quantized state Ψ_1^p into the anti-particle second quantized state with the properties as the starting state: the same d -momentum and the same eigen values of the Cartan sub algebra operators (S^{03} , S^{12} , S^{56}). One can easily check that the operation of $\mathcal{P}_\mathcal{H}^{(d-1)} e^{i\pi J_{12}} e^{i\pi J_{35}}$ on this anti-particle state (the hole in the Dirac sea) with the properties $S^{03} = \frac{i}{2}$, $S^{12} = \frac{1}{2}$, $S^{56} = \frac{1}{2}$ and the momentum ($|p^0|, 0, 0, |p^3|, 0, 0$) (manifesting in $e^{-ip^0x^0+ip^3x^3}$) transforms this anti-particle state into the anti-particle state $\overset{03}{(+i)} \overset{12}{(+)} \overset{56}{|} [-] e^{-ip^0x^0+ip^3x^3}$ applied on the vacuum, with the same spin and the same handedness in $d = (3 + 1)$ and the opposite "charge": $S^{56} = -\frac{1}{2}$, if we recognize the spin in $d = (5, 6)$ as the charge in $d = (3 + 1)$, as the starting second quantized state. But $\mathbb{C}_\mathcal{N} \psi_1^p = \overset{03}{(+i)} \overset{12}{[-]} \overset{56}{|} (+) e^{ip^0x^0-ip^3x^3}$ (solving the Weyl equation (6)) does not belong to the same Weyl representation as the starting state ψ_1^p . We can conclude that the charge conjugation operator

$$\mathbb{C}_\mathcal{N} \Psi_1^p (\mathbb{C}_\mathcal{N})^{-1} = \Psi_{\mathcal{N}1}^a. \quad (15)$$

is not a good symmetry.

Let us make the charge conjugation operation $\mathbb{C}_\mathcal{N}$ on the second quantized state \mathbf{u}_{1R} the corresponding single-particle state of which, on the top of the Dirac sea, is presented in the first line of Table II. This state has the quantum numbers of the right handed (with respect to $d = (3 + 1)$) u -quark (u_{1R}) with spin up, weak chargeless, with the hyper charge $\frac{2}{3}$ and the same electromagnetic charge, while its the second $SU(2)_{II}$ charge is $\frac{1}{2}$ and the colour charge is $(\frac{1}{2}, \frac{1}{(2\sqrt{3})})$. We find in Eq. (11) that $\mathcal{C}_\mathcal{H} u_{1R} = \overset{03}{(+i)} \overset{12}{[-]} \overset{56}{|} \overset{78}{[-]} \overset{9}{[-]} \overset{10}{[-]} \overset{11}{[-]} \overset{12}{[-]} \overset{13}{[-]} \overset{14}{[-]} e^{ip^0x^0-ip^3x^3}$.

To apply $\mathbb{C}_\mathcal{N}$ on u_{1R} we must, according to the definition in the first line of Eq. (13), multiply $\mathcal{C}_\mathcal{H} u_{1R}$ by $\mathcal{P}_\mathcal{H}^{(d-1)} e^{i\pi J_{12}} e^{i\pi J_{35}} e^{i\pi J_{79}} e^{i\pi J_{1113}}$. We end up with

$$\mathbb{C}_\mathcal{N} u_{1R} = \overset{03}{(+i)} \overset{12}{[-]} \overset{56}{|} \overset{78}{(+)} \overset{9}{(+)} \overset{10}{(+)} \overset{11}{(-)} \overset{12}{(-)} \overset{13}{(-)} \overset{14}{(-)} e^{ip^0x^0-ip^3x^3}. \quad (16)$$

The corresponding second quantized state is the hole in this single particle negative energy state in the Dirac sea (Fock space), which solves the Weyl equation for the negative energy state, but does not belong to the same Weyl representation, similarly as in the case with $d = (5 + 1)$. Although the corresponding second quantized state, that is the hole in the Dirac sea, $\mathbb{C}_\mathcal{N} \mathbf{u}_{1R} (\mathbb{C}_\mathcal{N})^{-1} (= \overset{03}{(+i)} \overset{12}{(+)} \overset{56}{|} \overset{78}{[-]} \overset{9}{[-]} \overset{10}{[-]} \overset{11}{[-]} \overset{12}{[-]} \overset{13}{[-]} \overset{14}{[-]} e^{-ip^0x^0+ip^3x^3} |vac\rangle)$ has the right charges, that is the opposite ones to those of the corresponding particle state, it is not a good symmetry. Again this is not within the same Weyl representation and correspondingly $\mathbb{C}_\mathcal{N}$ is not a good symmetry in $d = (13 + 1)$.

In all the spaces with $d = 2 \pmod{4}$ the charge conjugation operator \mathbb{C}_N is not a good symmetry within one Weyl representation: With a product of an odd number of γ^a it jumps out of the starting Weyl representation.

Parity symmetry \mathcal{P}_N .

\mathcal{P}_N (the third line in Eq. (13)) reflects only in the $d = (3 + 1)$ and multiplies spinors with γ^0 . It does not keep the transformed state within the same Weyl representation. In $d = (5 + 1)$ it transforms the single particle state Ψ_1^p into $[-i](+)^{03} | (+)^{12} e^{-ip^0 x^0 - ip^3 x^3} | vac >$, which is not within the same Weyl representation. In $d = (13 + 1)$ \mathcal{P}_N transforms u_{1R} into $[-i](+)^{03} | (+)^{12} (+)^{56} | vac >$, manifesting that \mathcal{P}_N is not a good symmetry in spaces with $d = 2 \pmod{4}$.

$\mathbb{C}_N \times \mathcal{P}_N$ symmetry.

Let us now check the $\mathbb{C}_N \mathcal{P}_N$ symmetry in $d = 2 \pmod{4}$. According to the first and the third line of Eq. (13) and to Eqs. (1, 3) it contains an even number of γ^a operators. Correspondingly the application of $\mathbb{C}_N \mathcal{P}_N$ on any state transforms the state again into the state within the same Weyl representation.

In $d = (5 + 1)$ we apply $\mathbb{C}_N \mathcal{P}_N$ on Ψ_1^p by applying \mathcal{P}_N on Ψ_{N1}^a . This is the single particle state ψ_4^p on the top of the Dirac sea, representing the hole in the state ψ_3^n in the Dirac sea. The state is within the same Weyl and solves the Weyl equation. The $\mathbb{C}_N \mathcal{P}_N$ manifests as a good symmetry.

Let in $d = (13 + 1)$ the operator $\mathbb{C}_N \mathcal{P}_N$ apply on the right handed \mathbf{u}_{1R} . One applies correspondingly \mathcal{P}_N on the second quantized anti-particle state ($\mathbb{C}_N \mathbf{u}_{1R}$) (which is the hole in the Dirac sea) with the same properties in $d = (3 + 1)$ as \mathbf{u}_{1R} (the single particle state created on the top of the Dirac sea) up to the "charges", which gained opposite values. The second quantized anti-particle state transforms, due to the application of \mathcal{P}_N , to the anti-particle state with the opposite (left) handedness in $d = (3 + 1)$. This anti-particle is recognized as a left handed weak chargeless anti- u -quark, of the anti-colour charge, belonging to the same Weyl representation (see the ref. [6], Table 4., line 35) and solving the Weyl equation (this state is $[-i](+)^{03} | [-]^{12} [-]^{56} | vac >$, manifesting that \mathcal{P}_N is not a good symmetry in spaces with $d = 2 \pmod{4}$).

Correspondingly $\mathbb{C}_N \mathcal{P}_N$ applied on ν_{1L} from Table III, seventh line, for example, transforms this left handed (in $d = (3 + 1)$) weak charged neutrino into the right handed (in $d = (3 + 1)$) weak anti-charged anti-colourless anti-neutrino (see the ref. [6], Table 5., line 61), belonging to the same Weyl representation and solving the Weyl equation.

$\mathbb{C}_N \mathcal{P}_N$ is a good symmetry in $d = 2(2n + 1) (= 2 \pmod{4})$ spaces.

Time reversal \mathcal{T}_N .

The application of the time reversal operator \mathcal{T}_N (the second equation in Eqs. (13)), constructed in spaces $d = 2 \pmod{4}$ out of an even number of γ^a operators, does keep the transformed state within the same Weyl representation.

Let us test on $d = (5 + 1)$ case first, applying \mathcal{T}_N on Ψ_1^p . The transformed state is Ψ_3^p from Table I: The state has the same handedness in $d = (3 + 1)$ as the starting state, the same S^{56} eigenvalue and opposite p^3 and S^{12} . Obviously \mathcal{T}_N is a good symmetry.

In the case of $d = (13 + 1)$ operator \mathcal{T}_N transforms u_{1R} with spin up from Table II into the state with spin down ($u_{2R} = \overset{03}{[-i]} \overset{12}{[-]} \overset{56}{|} \overset{78}{(+)(+)} \overset{9\ 10\ 11\ 12\ 13\ 14}{||} \overset{(+)(-)(-)}{e^{-ip^0x^0 - ip^3x^3}}$), keeping all the quantum numbers except eigenvalue of S^{03} and S^{12} the same and p^3 changes the sign. The state solves the Weyl equation.

\mathcal{T}_N is a good symmetry in any $d = 2 \pmod{4}$. It keeps states within the same Weyl representation and commutes with the operator $\gamma^a p_a$.

$\mathbb{C}_N \times \mathcal{P}_N \times \mathcal{T}_N$ **symmetry**.

In $d = (5 + 1)$ the operator $\mathbb{C}_N \mathcal{P}_N \mathcal{T}_N$ transforms Ψ_1^p from Table I, put on the top of the Dirac sea, into the positive energy anti-particle state Ψ_2^p on the top of the Dirac sea. It has an opposite handedness in $d = (1 + 3)$ and also the opposite spin and the opposite "charge".

In $d = (13 + 1)$ the operator $\mathbb{C}_N \mathcal{P}_N \mathcal{T}_N$ transforms u_{1R} from Table I, put on the top of the Dirac sea, into the positive energy state with the properties of \bar{u}_{1L} from the ref. [6], Table 4., line 36) (put on the top of the Dirac sea), weak chargeless, with the spin down and of the anti-colour charge.

$\mathbb{C}_N \mathcal{P}_N \mathcal{T}_N$ is a good symmetry, as it is expected to be.

IV. DISCUSSIONS AND CONCLUSIONS

We define in this paper the discrete symmetries, \mathbb{C}_N , \mathcal{P}_N and \mathcal{T}_N , (Eqs.(13)) in spaces $d = 2 \pmod{4}$, which lead to the experimentally observed symmetries, if the Kaluza-Klein-like theories, like it is the *spin-charge-family* theory of one of us (SNMB), are the right way to understand the assumptions of the *standard model*. Our $(\mathbb{C}_N, \mathcal{P}_N, \mathcal{T}_N)$ symmetries are rotated and reflected with respect to the symmetries as they would follow if extending the definition of the discrete symmetries from $d = (3 + 1)$ to any d belonging to $d = 2 \pmod{4}$. These discrete symmetries $(\mathbb{C}_H, \mathcal{P}_H$ and $\mathcal{T}_H)$, presented in Eqs. (1, 3), do not lead to the experimentally observed definitions, since if using \mathbb{C}_H on a second quantized state Ψ , the charge conjugated state has the same charge as the starting state. The proposed new discrete symmetries behave as they should. We do not study in this

paper the break of $\mathbb{C}_\mathcal{N}$ $\mathcal{P}_\mathcal{N}$ and $\mathcal{T}_\mathcal{N}$ symmetries.

We analyse properties of the proposed symmetries from the point of view of the observables in $d = (3 + 1)$. These discrete symmetries do not distinguish among families of fermions.

We illustrate our definition of the discrete symmetries on two cases: i. $d = (5 + 1)$ and ii. $d = (13 + 1)$. The first case is a toy model on which we show [4] that the Kaluza-Klein-like theories can lead in non-compact spaces to observable (almost massless) properties of fermions. We present in Table I one family of fermions of positive and negative energy states. There is, for example, a positive energy second quantized state ψ_1^p (the first state in Table I on the top of the Dirac sea), the $\mathcal{P}_\mathcal{N}$ transformed anti-particle state of which $\mathbb{C}_\mathcal{N} \mathcal{P}_\mathcal{N} \psi_1^p$ is the state with a positive energy and the same spin S^{12} , the opposite p^3 , the opposite handedness in $d = (3 + 1)$ and the opposite "charge" as the starting state and it is described by ψ_4^p , the fourth line in Table I on the top of the Dirac sea, which manifests as a hole in state ψ_3^n in the Dirac sea.

For the second illustration of the proposed discrete symmetries is taken the spinor representation of the *spin-charge-family* theory. We present in Table II the representation of quarks of particular colour charge and in Table III the representation of leptons. The rest of the whole one family representation can be found in the ref. [6]. The discrete symmetries proceed similarly to the case of $d = (5 + 1)$. Here fermions carry the experimentally recognized properties: $\mathbb{C}_\mathcal{N} \mathcal{P}_\mathcal{N}$ transforms the right handed u -quark with the spin up, weak chargeless and of the colour charge $(\frac{1}{2}, \frac{1}{2\sqrt{3}})$ and the hyper charge equal to $\frac{2}{3}$ into the left handed weak chargeless anti-quark with spin up and with the anti-colour charge $(-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$ and anti-hyper charge $-\frac{2}{3}$ (see also the ref. [6], table 4., line 1 and line 35). $\mathbb{C}_\mathcal{N} \mathcal{P}_\mathcal{N}$ transforms the weak charged $(\frac{1}{2})$ left handed neutrino, with spin up and colour chargeless into the right handed weak anti-charged $(-\frac{1}{2})$ anti-neutrino with the spin up, anti-colour chargeless (see also the ref. [6], Table 5., line 59).

The proposed discrete symmetries $\mathbb{C}_\mathcal{N}$, $\mathcal{P}_\mathcal{N}$ and $\mathcal{T}_\mathcal{N}$ have obviously the desired properties in the observable part of space.

Appendix: The technique for representing spinors [8, 10, 14, 15], a shortened version of the one presented in [8]

The technique [8, 10, 14, 15] can be used to construct a spinor basis for any dimension d and any signature in an easy and transparent way. Equipped with the graphic presentation of basic states, the technique offers an elegant way to see all the quantum numbers of states with respect to the Lorentz groups, as well as transformation properties of the states under any Clifford algebra

object.

The objects γ^a have properties

$$\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} I, \quad (\text{A.1})$$

for any d , even or odd. I is the unit element in the Clifford algebra.

The Clifford algebra objects S^{ab} close the algebra of the Lorentz group

$$\begin{aligned} S^{ab} &:= (i/4)(\gamma^a \gamma^b - \gamma^b \gamma^a), \\ \{S^{ab}, S^{cd}\}_- &= i(\eta^{ad} S^{bc} + \eta^{bc} S^{ad} - \eta^{ac} S^{bd} - \eta^{bd} S^{ac}). \end{aligned} \quad (\text{A.2})$$

The ‘‘Hermiticity’’ property for γ^a ’s: $\gamma^{a\dagger} = \eta^{aa} \gamma^a$ is assumed in order that γ^a are compatible with (A.1) and formally unitary, i.e. $\gamma^{a\dagger} \gamma^a = I$. One finds that $(S^{ab})^\dagger = \eta^{aa} \eta^{bb} S^{ab}$.

The Cartan subalgebra of the algebra is chosen in even dimensional spaces as follows

$$S^{03}, S^{12}, S^{56}, \dots, S^{d-1 d}, \quad \text{if } d = 2n \geq 4, \dots \quad (\text{A.3})$$

The choice for the Cartan subalgebra in $d < 4$ is straightforward. It is useful to define one of the Casimirs of the Lorentz group - the handedness Γ ($\{\Gamma, S^{ab}\}_- = 0$) in any d , for even dimensional spaces it follows

$$\Gamma^{(d)} := (i)^{d/2} \prod_a (\sqrt{\eta^{aa}} \gamma^a), \quad \text{if } d = 2n. \quad (\text{A.4})$$

The product of γ^a ’s in the ascending order with respect to the index a : $\gamma^0 \gamma^1 \dots \gamma^d$ is understood. It follows for any choice of the signature η^{aa} that $\Gamma^\dagger = \Gamma$, $\Gamma^2 = I$. For d even the handedness anticommutes with the Clifford algebra objects γ^a ($\{\gamma^a, \Gamma\}_+ = 0$) (while for d odd it commutes with γ^a ($\{\gamma^a, \Gamma\}_- = 0$)).

To make the technique simple the graphic presentation is introduced

$$\overset{ab}{(k)} := \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik} \gamma^b), \quad \overset{ab}{[k]} := \frac{1}{2}(1 + \frac{i}{k} \gamma^a \gamma^b), \quad (\text{A.5})$$

where $k^2 = \eta^{aa} \eta^{bb}$. One can easily check by taking into account the Clifford algebra relation (A.1) and the definition of S^{ab} (A.2) that if one multiplies from the left hand side by S^{ab} the Clifford algebra objects $\overset{ab}{(k)}$ and $\overset{ab}{[k]}$, it follows that

$$S^{ab} \overset{ab}{(k)} = \frac{1}{2} k \overset{ab}{(k)}, \quad S^{ab} \overset{ab}{[k]} = \frac{1}{2} k \overset{ab}{[k]}, \quad (\text{A.6})$$

which means that we get the same objects back multiplied by the constant $\frac{1}{2}k$. This also means that when $\overset{ab}{(k)}$ and $\overset{ab}{[k]}$ act from the left hand side on a vacuum state $|\psi_0\rangle$ the obtained states are

the eigenvectors of S^{ab} . One can further recognize that γ^a transform $\overset{ab}{(k)}$ into $\overset{ab}{[-k]}$, never to $\overset{ab}{[k]}$:

$$\gamma^a \overset{ab}{(k)} = \eta^{aa} \overset{ab}{[-k]}, \quad \gamma^b \overset{ab}{(k)} = -ik \overset{ab}{[-k]}, \quad \gamma^a \overset{ab}{[k]} = (-k), \quad \gamma^b \overset{ab}{[k]} = -ik\eta^{aa} \overset{ab}{(-k)}. \quad (\text{A.7})$$

From (A.7) it follows

$$\begin{aligned} S^{ac} \overset{ab}{(k)} \overset{cd}{(k)} &= -\frac{i}{2} \eta^{aa} \eta^{cc} \overset{ab}{[-k]} \overset{cd}{[-k]}, & S^{ac} \overset{ab}{[k]} \overset{cd}{[k]} &= \frac{i}{2} \overset{ab}{(-k)} \overset{cd}{(-k)}, \\ S^{ac} \overset{ab}{(k)} \overset{cd}{[k]} &= -\frac{i}{2} \eta^{aa} \overset{ab}{[-k]} \overset{cd}{(-k)}, & S^{ac} \overset{ab}{[k]} \overset{cd}{(k)} &= \frac{i}{2} \eta^{cc} \overset{ab}{(-k)} \overset{cd}{[-k]}. \end{aligned} \quad (\text{A.8})$$

Let us deduce some useful relations

$$\begin{aligned} \overset{ab}{(k)} \overset{ab}{(k)} &= 0, & \overset{ab}{(k)} \overset{ab}{(-k)} &= \eta^{aa} \overset{ab}{[k]}, & \overset{ab}{[k]} \overset{ab}{[k]} &= \overset{ab}{[k]}, & \overset{ab}{[k]} \overset{ab}{[-k]} &= 0, \\ \overset{ab}{(k)} \overset{ab}{[k]} &= 0, & \overset{ab}{[k]} \overset{ab}{(k)} &= \overset{ab}{(k)}, & \overset{ab}{(k)} \overset{ab}{[-k]} &= \overset{ab}{(k)}, & \overset{ab}{[k]} \overset{ab}{(-k)} &= 0. \end{aligned} \quad (\text{A.9})$$

We recognize in the first equation of the first line and the third equation of the first line the demonstration of the nilpotent and the projector character of the Clifford algebra objects $\overset{ab}{(k)}$ and $\overset{ab}{[k]}$, respectively. Recognizing that $\overset{ab}{(k)} = \eta^{aa} \overset{ab}{(-k)}$, $\overset{ab}{[k]} = \overset{ab}{[k]}$, a vacuum state $|\psi_0\rangle$ can be defined so that it follows

$$\langle \overset{ab}{(k)} \overset{ab}{(k)} \rangle = 1, \quad \langle \overset{ab}{[k]} \overset{ab}{[k]} \rangle = 1. \quad (\text{A.10})$$

Taking into account the above equations it is easy to find a Weyl spinor irreducible representation for d -dimensional space.

For d even we simply make a starting state as a product of $d/2$, let us say, only nilpotents $\overset{ab}{(k)}$, one for each S^{ab} of the Cartan subalgebra elements (A.3), applying it on an (unimportant) vacuum state. Then the generators S^{ab} , which do not belong to the Cartan subalgebra, being applied on the starting state from the left, generate all the members of one Weyl spinor.

$$\begin{aligned} &\overset{0d}{(k_{0d})} \overset{12}{(k_{12})} \overset{35}{(k_{35})} \cdots \overset{d-1}{(k_{d-1})} \overset{d-2}{(k_{d-2})} \psi_0 \\ &\overset{0d}{[-k_{0d}]} \overset{12}{[-k_{12}]} \overset{35}{(k_{35})} \cdots \overset{d-1}{(k_{d-1})} \overset{d-2}{(k_{d-2})} \psi_0 \\ &\overset{0d}{[-k_{0d}]} \overset{12}{(k_{12})} \overset{35}{[-k_{35}]} \cdots \overset{d-1}{(k_{d-1})} \overset{d-2}{(k_{d-2})} \psi_0 \\ &\vdots \\ &\overset{0d}{[-k_{0d}]} \overset{12}{(k_{12})} \overset{35}{(k_{35})} \cdots \overset{d-1}{[-k_{d-1}]} \overset{d-2}{(k_{d-2})} \psi_0 \\ &\overset{0d}{(k_{0d})} \overset{12}{[-k_{12}]} \overset{35}{[-k_{35}]} \cdots \overset{d-1}{(k_{d-1})} \overset{d-2}{(k_{d-2})} \psi_0 \\ &\vdots \end{aligned} \quad (\text{A.11})$$

All the states have the handedness Γ , since $\{\Gamma, S^{ab}\}_- = 0$. States, belonging to one multiplet with respect to the group $SO(q, d - q)$, that is to one irreducible representation of spinors (one Weyl spinor), can have any phase. We made a choice of the simplest one, taking all phases equal to one.

The above graphic representation demonstrates that for d even all the states of one irreducible Weyl representation of a definite handedness follow from a starting state, which is, for example, a product of nilpotents $(k_{ab})^{ab}$, by transforming all possible pairs of $(k_{ab})^{ab}(k_{mn})^{mn}$ into $[-k_{ab}]^{ab}[-k_{mn}]^{mn}$. There are $S^{am}, S^{an}, S^{bm}, S^{bn}$, which do this. The procedure gives $2^{(d/2-1)}$ states. A Clifford algebra object γ^a being applied from the left hand side, transforms a Weyl spinor of one handedness into a Weyl spinor of the opposite handedness. Both Weyl spinors form a Dirac spinor.

We shall speak about left handedness when $\Gamma = -1$ and about right handedness when $\Gamma = 1$.

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